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AUTHOR(S):

Ohta, Masahito

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# Stability of solitary waves for coupled Klein-Gordon-Schrödinger equations in one space dimension

Masahito Ohta (太田 雅人)

Faculty of Engineering, Shizuoka University, Hamamatsu 432-8561, Japan

Email:tsmoota@eng.shizuoka.ac.jp

## 1 Introduction and Main Result

In this note we consider the stability of solitary wave solutions for the Yukawa coupled Klein-Gordon-Schrödinger equations in one space dimension:

$$i\partial_t u + \partial_x^2 u = uv, \quad (t, x) \in \mathbf{R} \times \mathbf{R}, \quad (1.1)$$

$$\partial_t^2 v - \partial_x^2 v + v = -|u|^2, \quad (t, x) \in \mathbf{R} \times \mathbf{R}. \quad (1.2)$$

Here,  $u = u(t, x)$  and  $v = v(t, x)$  describe a complex scalar nucleon field and a real scalar meson field, respectively (see Fukuda and M. Tsutsumi [3] and Yukawa [11]). In Section 5 of [3], Fukuda and M. Tsutsumi showed that (1.1)–(1.2) admits the following two types of exact solitary wave solutions (1.3)–(1.4) and (1.5)–(1.6):

(I) when  $\lambda^2 < 1$  and  $\mu = \lambda^2/4 + 1/(1 - \lambda^2)$ ,

$$u(t, x) = \frac{3}{2\sqrt{1 - \lambda^2}} \operatorname{sech}^2 \left( \frac{x - \lambda t}{2\sqrt{1 - \lambda^2}} \right) \exp[i\mu t + i(\lambda/2)(x - \lambda t)], \quad (1.3)$$

$$v(t, x) = -\frac{3}{2(1 - \lambda^2)} \operatorname{sech}^2 \left( \frac{x - \lambda t}{2\sqrt{1 - \lambda^2}} \right), \quad (1.4)$$

(II) when  $\lambda^2 = 1$  and  $\mu > 1/4$ ,

$$u(t, x) = \sqrt{2(\mu - 1/4)} \operatorname{sech} \left( \sqrt{\mu - 1/4} (x - \lambda t) \right) \times \exp[i\mu t + i(\lambda/2)(x - \lambda t)], \quad (1.5)$$

$$v(t, x) = -2(\mu - 1/4) \operatorname{sech}^2 \left( \sqrt{\mu - 1/4} (x - \lambda t) \right), \quad (1.6)$$

and they proposed a problem of whether the solitary wave solutions are stable or not. The purpose of this note is to give a partial answer to the problem.

To explain our results precisely, we prepare some function spaces and functionals. Let  $X = H^1(\mathbf{R}; \mathbf{C}) \times H^1(\mathbf{R}; \mathbf{R}) \times L^2(\mathbf{R}; \mathbf{R})$  be a real Hilbert space with the inner product

$$\begin{aligned} ((u, v, w), (\psi, \phi, \varphi))_X &= 2 \operatorname{Re} \int_{\mathbf{R}} \left( \partial_x u(x) \overline{\partial_x \psi(x)} + u(x) \overline{\psi(x)} \right) dx \\ &\quad + \int_{\mathbf{R}} (\partial_x v(x) \partial_x \phi(x) + v(x) \phi(x) + w(x) \varphi(x)) dx \end{aligned}$$

for  $(u, v, w), (\psi, \phi, \varphi) \in X$ . Then, (1.1)–(1.2) is written as abstract Hamiltonian system in the form

$$\frac{d}{dt} \vec{u}(t) = J E'(\vec{u}(t)), \quad (1.7)$$

where

$$\vec{u}(t) = \begin{pmatrix} u(t) \\ v(t) \\ w(t) \end{pmatrix} \in X, \quad J = \begin{pmatrix} -i & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix},$$

and  $E$  is the energy functional on  $X$  defined by

$$E(u, v, w) = \int_{\mathbf{R}} \left\{ |\partial_x u|^2 + |u|^2 v + \frac{1}{2} (w^2 + (\partial_x v)^2 + v^2) \right\} dx. \quad (1.8)$$

The energy functional  $E(\vec{u})$  is invariant under the action  $T(\alpha, \beta)$  of the group defined by

$$T(\alpha, \beta)(u, v, w)(x) = (e^{i\beta} u(x + \alpha), v(x + \alpha), w(x + \alpha))$$

for  $\alpha, \beta \in \mathbf{R}$  and  $(u, v, w) \in X$ . Associated with the group action  $T(\alpha, \beta)$ , we define two conserved functionals on  $X$ , the momentum  $P$  and the charge  $Q$ , by

$$P(u, v, w) = \int_{\mathbf{R}} (i\bar{u} \partial_x u + w \partial_x v) dx, \quad (1.9)$$

$$Q(u, v, w) = \int_{\mathbf{R}} |u|^2 dx. \quad (1.10)$$

The following global existence of solutions of the Cauchy problem for (1.7) is known.

**Proposition 1.1** For any  $\vec{u}_0 \in X$  there exists a unique solution  $\vec{u} \in C(\mathbf{R}; X)$  of (1.7) with  $\vec{u}(0) = \vec{u}_0$  satisfying

$$E(\vec{u}(t)) = E(\vec{u}_0), \quad P(\vec{u}(t)) = P(\vec{u}_0), \quad Q(\vec{u}(t)) = Q(\vec{u}_0), \quad t \in \mathbf{R}. \quad (1.11)$$

For  $\lambda, \mu \in \mathbf{R}$ , we put

$$S_{\lambda, \mu}(\vec{u}) = E(\vec{u}) + \lambda P(\vec{u}) + \mu Q(\vec{u}), \quad \vec{u} \in X.$$

Then, we have

$$S_{\lambda, \mu}(u, v, w) = \int_{\mathbf{R}} \left\{ \left| \partial_x (e^{-i\lambda x/2} u) \right|^2 + (\mu - \lambda^2/4) |u|^2 + |u|^2 v + \frac{1}{2} \left( (w + \lambda \partial_x v)^2 + (1 - \lambda^2)(\partial_x v)^2 + v^2 \right) \right\} dx, \quad (1.12)$$

and  $T(\lambda t, \mu t) \vec{\psi}_{\lambda, \mu}$  is a solution of (1.7) if  $\vec{\psi}_{\lambda, \mu}$  is a solution of  $S'_{\lambda, \mu}(\vec{\psi}) = 0$ . Note that  $S'_{\lambda, \mu}(\psi, \phi, \varphi) = E'(\psi, \phi, \varphi) + \lambda P'(\psi, \phi, \varphi) + \mu Q'(\psi, \phi, \varphi) = 0$  is equivalent to

$$-\partial_x^2 \psi + i\lambda \partial_x \psi + \mu \psi + \psi \phi = 0, \quad (1.13)$$

$$-(1 - \lambda^2) \partial_x^2 \phi + \phi + |\psi|^2 = 0, \quad (1.14)$$

$$\varphi + \lambda \partial_x \phi = 0, \quad (1.15)$$

and also that by  $\psi(x) = e^{i(\lambda/2)x} \tilde{\psi}(x)$ , (1.13) is transformed into

$$-\partial_x^2 \tilde{\psi} + (\mu - \lambda^2/4) \tilde{\psi} + \tilde{\psi} \phi = 0. \quad (1.16)$$

Thus, when  $\lambda^2 < 1$  and  $\mu = \lambda^2/4 + 1/(1 - \lambda^2)$ , if we put

$$\psi_{\lambda, \mu}(x) = \frac{3}{2\sqrt{1 - \lambda^2}} \operatorname{sech}^2 \left( \frac{x}{2\sqrt{1 - \lambda^2}} \right) \exp[i(\lambda/2)x], \quad (1.17)$$

$$\phi_{\lambda, \mu}(x) = -\frac{3}{2(1 - \lambda^2)} \operatorname{sech}^2 \left( \frac{x}{2\sqrt{1 - \lambda^2}} \right), \quad (1.18)$$

$$\varphi_{\lambda, \mu}(x) = -\lambda \partial_x \phi_{\lambda, \mu}(x), \quad (1.19)$$

then  $\vec{\psi}_{\lambda, \mu} = (\psi_{\lambda, \mu}, \phi_{\lambda, \mu}, \varphi_{\lambda, \mu})$  is a solution of  $S'_{\lambda, \mu}(\vec{\psi}) = 0$ , and when  $\lambda^2 = 1$  and  $\mu > 1/4$ , if we put

$$\psi_{\lambda, \mu}(x) = \sqrt{2(\mu - 1/4)} \operatorname{sech} \left( \sqrt{\mu - 1/4} x \right) \exp[i(\lambda/2)x], \quad (1.20)$$

$$\phi_{\lambda, \mu}(x) = -2(\mu - 1/4) \operatorname{sech}^2 \left( \sqrt{\mu - 1/4} x \right), \quad (1.21)$$

$$\varphi_{\lambda, \mu}(x) = -\lambda \partial_x \phi_{\lambda, \mu}(x), \quad (1.22)$$

then  $\vec{\psi}_{\lambda,\mu} = (\psi_{\lambda,\mu}, \phi_{\lambda,\mu}, \varphi_{\lambda,\mu})$  is a solution of  $S'_{\lambda,\mu}(\vec{\psi}) = 0$ .

**Definition.** We say that a subset  $\Sigma$  of  $X$  is *stable* if for any  $\varepsilon > 0$  there exists a  $\delta > 0$  with the following property. If  $\vec{u}_0 \in X$  satisfies  $\inf\{\|\vec{u}_0 - \vec{\psi}\|_X : \vec{\psi} \in \Sigma\} < \delta$ , then the solution  $\vec{u}(t)$  of (1.7) with  $\vec{u}(0) = \vec{u}_0$  exists for all  $t \in \mathbf{R}$  and satisfies

$$\sup_{t \in \mathbf{R}} \inf\{\|\vec{u}(t) - \vec{\psi}\|_X : \vec{\psi} \in \Sigma\} < \varepsilon.$$

Moreover, let  $\vec{\psi}_{\lambda,\mu}$  be a solution of  $S'_{\lambda,\mu}(\vec{\psi}) = 0$ . We say that the solitary wave solution  $T(\lambda t, \mu t)\vec{\psi}_{\lambda,\mu}$  is *stable* if  $\{T(\alpha, \beta)\vec{\psi}_{\lambda,\mu} : \alpha, \beta \in \mathbf{R}\}$  is stable.

We are now in a position to state our main result in this note.

**Theorem II.** *Let  $\lambda^2 = 1$  and  $\mu > 1/4$ . Then, the solitary wave solution  $T(\lambda t, \mu t)\vec{\psi}_{\lambda,\mu}$  given by (1.20)–(1.22) is stable for any  $\mu > 1/4$ .*

**Remark.** In my lecture at the conference, I announced that when  $\lambda^2 < 1$  and  $\mu = \lambda^2/4 + 1/(1 - \lambda^2)$ , the solitary wave solution  $T(\lambda t, \mu t)\vec{\psi}_{\lambda,\mu}$  given by (1.17)–(1.19) is stable if  $\lambda^2$  is sufficiently close to 1. However, after the conference, I found a mistake in the proof. So, the stability of  $T(\lambda t, \mu t)\vec{\psi}_{\lambda,\mu}$  given by (1.17)–(1.19) seems to be still an open problem.

## 2 Proof of Theorem II

In this section, we give the proof of Theorem II, basically along the argument in [7].

When  $\lambda^2 = 1$  and  $\mu > 1/4$ , we obtain the following basic identity from (1.12).

$$\begin{aligned} S_{\lambda,\mu}(u, v, w) &= \int_{\mathbf{R}} \left\{ \left| \partial_x(e^{-i\lambda x/2}u) \right|^2 + (\mu - 1/4)|u|^2 - \frac{1}{2}|u|^4 \right\} dx \\ &\quad + \frac{1}{2} \int_{\mathbf{R}} \left\{ (w + \lambda \partial_x v)^2 + (|u|^2 + v)^2 \right\} dx. \end{aligned} \quad (2.1)$$

Associated with the identity (2.1), we define for  $\rho > 0$

$$S_{\lambda,\mu}^0(u) = \int_{\mathbf{R}} \left\{ \left| \partial_x(e^{-i\lambda x/2}u) \right|^2 + (\mu - 1/4)|u|^2 - \frac{1}{2}|u|^4 \right\} dx, \quad (2.2)$$

$$Q^0(u) = \int_{\mathbf{R}} |u|^2 dx, \quad (2.3)$$

$$I^0(\rho) = \inf\{S_{\lambda,\mu}^0(u) : u \in H^1(\mathbf{R}), Q^0(u) = \rho\}, \quad (2.4)$$

$$\Sigma^0(\rho) = \{u \in H^1(\mathbf{R}) : S_{\lambda,\mu}^0(u) = I^0(\rho), Q^0(u) = \rho\}. \quad (2.5)$$

For  $\alpha, \beta \in \mathbf{R}$  and  $u \in L^2(\mathbf{R})$ , we define

$$T_1(\alpha)u(x) = u(x + \alpha), \quad T_2(\beta)u(x) = e^{i\beta}u(x).$$

**Lemma 2.1** *Assume that  $\lambda^2 = 1$  and  $\mu > 1/4$ . Let  $\psi_{\lambda,\mu}$  be the function defined by (1.20). Then, we have*

$$\Sigma^0(\rho(\mu)) = \{T_1(\alpha)T_2(\beta)\psi_{\lambda,\mu} : \alpha, \beta \in \mathbf{R}\},$$

where  $\rho(\mu) = Q^0(\psi_{\lambda,\mu}) = 4\sqrt{\mu - 1/4}$ .

**Lemma 2.2** *Let  $\rho > 0$ . If  $\{u_j\} \subset H^1(\mathbf{R})$  satisfies  $S_{\lambda,\mu}^0(u_j) \rightarrow I^0(\rho)$  and  $Q^0(u_j) \rightarrow \rho$ , then there exist  $\{\alpha_j\} \subset \mathbf{R}$ , a subsequence of  $\{T_1(\alpha_j)u_j\}$  (we still denote it by the same letter) and  $\psi \in \Sigma^0(\rho)$  such that*

$$T_1(\alpha_j)u_j \rightarrow \psi \quad \text{strongly in } H^1(\mathbf{R}).$$

Lemma 2.2 is proved by using the concentration compactness method introduced by Lions [6]. For the proofs of Lemmas 2.1 and 2.2, see Cazenave and Lions [1]. From Lemmas 2.1 and 2.2 and the conservation laws, one can show the stability of solitary wave solutions for the single nonlinear Schrödinger equation (for details, see [1]).

Following the idea by Cazenave and Lions [1], we consider the following minimization problem:

$$I(\rho) = \inf\{S_{\lambda,\mu}(\vec{u}) : \vec{u} \in X, Q(\vec{u}) = \rho\}, \quad (2.6)$$

$$\Sigma(\rho) = \{\vec{u} \in X : S_{\lambda,\mu}(\vec{u}) = I(\rho), Q(\vec{u}) = \rho\}. \quad (2.7)$$

From Lemma 2.1 and (2.1), we have

**Lemma 2.3** *Assume that  $\lambda^2 = 1$  and  $\mu > 1/4$ . For any  $\rho > 0$ , we have  $I(\rho) = I^0(\rho)$  and*

$$\Sigma(\rho) = \{\vec{\psi} = (\psi, \phi, \varphi) : \psi \in \Sigma^0(\rho), \phi = -|\psi|^2, \varphi = -\lambda\partial_x\psi\}. \quad (2.8)$$

Moreover, let  $\vec{\psi}_{\lambda,\mu} = (\psi_{\lambda,\mu}, \phi_{\lambda,\mu}, \varphi_{\lambda,\mu})$  be the vector in  $X$  given by (1.20)–(1.22). Then, we have

$$\Sigma(\rho(\mu)) = \{T(\alpha, \beta)\vec{\psi}_{\lambda,\mu} : \alpha, \beta \in \mathbf{R}\}, \quad (2.9)$$

where  $\rho(\mu) = Q(\vec{\psi}_{\lambda,\mu}) = 4\sqrt{\mu - 1/4}$ .

**Proof.** First, we note that  $S_{\lambda,\mu}^0(u) \leq S_{\lambda,\mu}(\vec{u})$  holds for all  $\vec{u} = (u, v, w) \in X$ , so that we have  $I^0(\rho) \leq I(\rho)$ . We put

$$\Sigma_1(\rho) = \left\{ \vec{\psi} = (\psi, \phi, \varphi) : \psi \in \Sigma^0(\rho), \phi = -|\psi|^2, \varphi = -\lambda \partial_x \phi \right\}.$$

If  $\vec{\psi} = (\psi, \phi, \varphi) \in \Sigma_1(\rho)$ , then we have  $Q(\vec{\psi}) = Q^0(\psi) = \rho$  and

$$I(\rho) \leq S_{\lambda,\mu}(\vec{\psi}) = S_{\lambda,\mu}^0(\psi) = I^0(\rho) \leq I(\rho).$$

Thus, we have  $I(\rho) = I^0(\rho)$  and  $\vec{\psi} \in \Sigma(\rho)$ . Conversely, if  $\vec{\psi} = (\psi, \phi, \varphi) \in \Sigma(\rho)$ , then we have  $Q(\vec{\psi}) = Q^0(\psi) = \rho$  and

$$I^0(\rho) \leq S_{\lambda,\mu}^0(\psi) \leq S_{\lambda,\mu}(\vec{\psi}) = I(\rho) = I^0(\rho).$$

Thus, we have  $\vec{\psi} \in \Sigma_1(\rho)$ . Hence, we obtain (2.8). (2.9) follows from Lemma 2.1 and (2.8). This completes the proof.  $\square$

**Lemma 2.4** *Let  $\rho > 0$  and  $\vec{\psi}_0 \in \Sigma(\rho)$ . If  $\{\vec{u}_j\} = \{(u_j, v_j, w_j)\} \subset X$  satisfies*

$$E(\vec{u}_j) \rightarrow E(\vec{\psi}_0), \quad P(\vec{u}_j) \rightarrow P(\vec{\psi}_0), \quad Q(\vec{u}_j) \rightarrow Q(\vec{\psi}_0). \quad (2.10)$$

*then there exist  $\{\alpha_j\} \subset \mathbf{R}$ , a subsequence of  $\{T(\alpha_j, 0)\vec{u}_j\}$  (we still denote it by the same letter) and  $\vec{\psi}_1 = (\psi_1, \phi_1, \varphi_1) \in \Sigma(\rho)$  such that*

$$T(\alpha_j, 0)\vec{u}_j \rightarrow \vec{\psi}_1 \quad \text{strongly in } X.$$

**Proof.** First, we note that by the Gagliardo-Nirenberg-Sobolev inequality and (2.10), we see that  $\{\vec{u}_j\}$  is a bounded sequence in  $X$ , and

$$S_{\lambda,\mu}(\vec{u}_j) = E(\vec{u}_j) + \lambda P(\vec{u}_j) + \mu Q(\vec{u}_j) \rightarrow S_{\lambda,\mu}(\vec{\psi}_0) = I(\rho). \quad (2.11)$$

Since we have

$$I(\rho) = I^0(\rho) \leq S_{\lambda,\mu}^0(u_j) \leq S_{\lambda,\mu}(\vec{u}_j),$$

it follows from (2.11) and (2.10) that

$$S_{\lambda,\mu}^0(u_j) \rightarrow I^0(\rho), \quad Q^0(u_j) = Q(\vec{u}_j) \rightarrow \rho.$$

Thus, by Lemma 2.2, there exist  $\{\alpha_j\} \subset \mathbf{R}$  and a subsequence of  $\{T_1(\alpha_j)u_j\}$  (we still denote it by the same letter) and  $\tilde{\psi} \in \Sigma^0(\rho)$  such that

$$T_1(\alpha_j)u_j \rightarrow \tilde{\psi} \quad \text{strongly in } H^1(\mathbf{R}). \quad (2.12)$$

Since  $\{\vec{u}_j\}$  is bounded in  $X$ , so is  $\{T(\alpha_j, 0)\vec{u}_j\}$ . Thus, there exists a subsequence  $\{\vec{u}_j^1\} = \{(u_j^1, v_j^1, w_j^1)\}$  of  $\{T(\alpha_j, 0)\vec{u}_j\}$  and  $\vec{\psi}_1 = (\psi_1, \phi_1, \varphi_1) \in X$  such that

$$\vec{u}_j^1 \rightharpoonup \vec{\psi}_1 \quad \text{weakly in } X. \quad (2.13)$$

By (2.12) and (2.13), we have  $\psi_1 = \tilde{\psi} \in \Sigma^0(\rho)$  and

$$u_j^1 \rightarrow \psi_1 \quad \text{strongly in } H^1(\mathbf{R}). \quad (2.14)$$

Moreover, from (2.1) and (2.14), we have

$$|u_j^1|^2 + v_j^1 \rightarrow 0 \quad \text{strongly in } L^2(\mathbf{R}), \quad (2.15)$$

$$w_j^1 + \lambda \partial_x v_j^1 \rightarrow 0 \quad \text{strongly in } L^2(\mathbf{R}). \quad (2.16)$$

From (2.13)–(2.16), we have

$$v_j^1 \rightarrow \phi_1 = -|\psi_1|^2 \quad \text{strongly in } L^2(\mathbf{R}), \quad (2.17)$$

$$\partial_x v_j^1 \rightharpoonup \partial_x \phi_1 \quad \text{weakly in } L^2(\mathbf{R}), \quad (2.18)$$

$$w_j^1 \rightharpoonup \varphi_1 = -\lambda \partial_x \phi_1 \quad \text{weakly in } L^2(\mathbf{R}). \quad (2.19)$$

Since  $\psi_1 \in \Sigma^0(\rho)$ ,  $\phi_1 = -|\psi_1|^2$  and  $\varphi_1 = -\lambda \partial_x \phi_1$ , it follows from Lemma 2.3 that  $\vec{\psi}_1 = (\psi_1, \phi_1, \varphi_1) \in \Sigma(\rho)$ . Finally, we have to show the strong convergence of  $\{\partial_x v_j^1\}$  and  $\{w_j^1\}$  in  $L^2(\mathbf{R})$ . By the definition (1.9) and the convergences in (2.10) and (2.14), we have

$$\begin{aligned} \int_{\mathbf{R}} w_j^1 \partial_x v_j^1 dx &= P(\vec{u}_j^1) - \int_{\mathbf{R}} i \overline{u_j^1} \partial_x u_j^1 dx \\ &= P(\vec{u}_j) - \int_{\mathbf{R}} i \overline{u_j^1} \partial_x u_j^1 dx \rightarrow P(\vec{\psi}_0) - \int_{\mathbf{R}} i \overline{\psi_1} \partial_x \psi_1 dx. \end{aligned}$$

Since  $\vec{\psi}_0, \vec{\psi}_1 \in \Sigma(\rho)$ , from Lemma 2.3, we have  $P(\vec{\psi}_0) = P(\vec{\psi}_1)$ . Thus, we have

$$\int_{\mathbf{R}} w_j^1 \partial_x v_j^1 dx \rightarrow \int_{\mathbf{R}} \varphi_1 \partial_x \phi_1 dx. \quad (2.20)$$



Therefore, by (2.16) and (2.18)–(2.20), we have

$$\begin{aligned}
\lambda^2 \|\partial_x \phi_1\|_{L^2}^2 + \|\varphi_1\|_{L^2}^2 &\leq \liminf_{j \rightarrow \infty} \left( \lambda^2 \|\partial_x v_j^1\|_{L^2}^2 + \|w_j^1\|_{L^2}^2 \right) \\
&= \liminf_{j \rightarrow \infty} \left( \|w_j^1 + \lambda \partial_x v_j^1\|_{L^2}^2 - 2\lambda \int_{\mathbf{R}} w_j^1 \partial_x v_j^1 dx \right) \\
&= -2\lambda \int_{\mathbf{R}} \varphi_1 \partial_x \phi_1 dx = \lambda^2 \|\partial_x \phi_1\|_{L^2}^2 + \|\varphi_1\|_{L^2}^2.
\end{aligned}$$

Here we have used the fact that  $\varphi_1 + \lambda \partial_x \phi_1 = 0$  in the last identity. Hence, we obtain

$$\partial_x v_j^1 \rightarrow \partial_x \phi_1 \quad \text{strongly in} \quad L^2(\mathbf{R}), \quad (2.21)$$

$$w_j^1 \rightarrow \varphi_1 \quad \text{strongly in} \quad L^2(\mathbf{R}). \quad (2.22)$$

This completes the proof.  $\square$

**Proof of Theorem II.** By Lemma 2.3, it is enough to show that  $\Sigma(\rho)$  is stable for any  $\rho > 0$ . We prove it by contradiction. Suppose that  $\Sigma(\rho)$  is not stable. Then, by the definition, there exist a constant  $\varepsilon_0 > 0$  and sequences  $\{\vec{u}_{0j}\} \subset X$  and  $\{t_j\} \subset \mathbf{R}$  such that

$$\liminf_{j \rightarrow \infty} \{\|\vec{u}_{0j} - \vec{\psi}\|_X : \vec{\psi} \in \Sigma(\rho)\} = 0 \quad (2.23)$$

and

$$\inf\{\|\vec{u}_j(t_j) - \vec{\psi}\|_X : \vec{\psi} \in \Sigma(\rho)\} \geq \varepsilon_0, \quad (2.24)$$

where  $\vec{u}_j(t)$  is the solution of (1.7) with  $\vec{u}_j(0) = \vec{u}_{0j}$ . By (2.23) and the conservation laws (1.11), we have

$$\begin{aligned}
E(\vec{u}_j(t_j)) &= E(\vec{u}_{0j}) \rightarrow E(\vec{\psi}_0), \\
P(\vec{u}_j(t_j)) &= P(\vec{u}_{0j}) \rightarrow P(\vec{\psi}_0), \\
Q(\vec{u}_j(t_j)) &= Q(\vec{u}_{0j}) \rightarrow Q(\vec{\psi}_0)
\end{aligned}$$

for some  $\vec{\psi}_0 \in \Sigma(\rho)$ . Thus, by Lemma 2.4, there exist  $\{\alpha_j\} \subset \mathbf{R}$ , a subsequence of  $\{T(\alpha_j, 0)\vec{u}_j(t_j)\}$  (we still denote it by the same letter) and  $\vec{\psi}_1 \in \Sigma(\rho)$  such that

$$T(\alpha_j, 0)\vec{u}_j(t_j) \rightarrow \vec{\psi}_1 \quad \text{strongly in} \quad X.$$

However, this contradicts (2.24). Hence,  $\Sigma(\rho)$  is stable.  $\square$

### 3 Concluding Remarks

When  $\lambda^2 > 1$  and  $\mu = \lambda^2/4 + 1/(2(\lambda^2 - 1))$ , (1.1)–(1.2) also admit the following exact solitary wave solutions (see [10]):

$$u(t, x) = \frac{3}{\sqrt{\lambda^2 - 1}} \operatorname{sech} \left( \frac{x - \lambda t}{\sqrt{2(\lambda^2 - 1)}} \right) \tanh \left( \frac{x - \lambda t}{\sqrt{2(\lambda^2 - 1)}} \right) \times \exp[i\mu t + i(\lambda/2)(x - \lambda t)], \quad (3.1)$$

$$v(t, x) = -\frac{3}{\lambda^2 - 1} \operatorname{sech}^2 \left( \frac{x - \lambda t}{\sqrt{2(\lambda^2 - 1)}} \right). \quad (3.2)$$

The variational characterizations and the stability problem for (3.1)–(3.2) seem to be open problems.

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